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THE NECESSARY AND SUFFICIENT CONDITIONS OF STABILITY IN THE LARGE*

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Stability in the large on stationary sets is considered. A definition of stability which extends that of Kamenkov /1/ over a fairly wide class of sets is introduced. The derived necessary and sufficient conditions of stability are, unlike in /1—9/, constructively verifiable on a computer in the sense of probability and make possible investigation of stability of systems of various physical properties.

It is shown that the applied here algorithm converges with respect to probability to the solution of the problem. The necessary condition of stability, and of instability on some particular assumptions are obtained. They can be analytically verified. A fairly comprehensive survey of publications on investigations of stability in the large appeared in /2,10/.

1. Consider the equations of perturbed motion of the form

$$dx/dt = f(x) \tag{1.1}$$

where x is an n-dimensional vector of space R^n, and $f(x)$ is a continuous vector function ($f(x) \in C(R^n)$). We assume that $f(x)$ is such that the solution of system (1.1) satisfies the following conditions (F): 1) the solution exists and is unique for any initial condition $x_0 = x(t_0) \in R^n$; 2) the solution is continuous with respect to initial conditions x_0. The conditions of fulfillment of (F) appear, for instance, in /11/.

We shall use the following notation: θ for the coordinate origin of space R^n, $\text{mes}_{R^n} G$ for the n-dimensional volume of set G (the Lebesgue measure in space R^n is the product of n Lebesgue measures in R^1), E_W is the set of values of function $W(x)$, dG^- is the outer boundary of set G, and

$$S_R = \{x \mid \sum_{i=1}^n x_i^2 = R\}, \quad \|W\|_{C(G)} = \sup_{x \in G} |W(x)|, \quad \rho(x, G) = \inf_{y \in G} \Big[\sum_{i=1}^n (y_i - x_i)^2\Big]^{1/2}$$

Below, space will be understood to mean a set homeomorphic to a sphere.

We introduce the class of sets K whose every element has the following properties: ($\forall G \in K$) G is closed and bounded, $\text{mes}_{R^n} G > 0$, $\theta \in G$, ∂G^- is a surface, set G contains a point for which the homothetic transformation with center at that point in some neighborhood of dG^- converts set dG^- into nonintersecting surfaces.

The last condition is satisfied in the case of a convex set G.

We assume without loss of generality that the point under consideration coincides with θ, since the latter can always be brought to that point.

We shall call the set $\text{int}_1(\partial G^-)$ bounded by the outer boundary of set $G \in K$ the largest connected set whose external boundary is ∂G^-. We denote by $\text{int}(\partial G^-) = \text{int}_1(\partial G^-) \setminus \partial G^-$. The largest set H is understood to be that which comprises any set posessing the required properties. The largest connected set exists by virtue of $G \in K$.

2. Let us formulate the definition of stability (in the large) on set $G \in K$, with measure $\text{mes}_{R^n} G$ not necessarily small.

Definition 1. System (1.1) is stable on set G in the meaning of this definition, if

$$(\forall x(t_0))\, x(t_0) \in G \Rightarrow (\forall t \geqslant t_0)\, x(t, t_0, x(t_0)) \in \text{int}_1(\partial G^-) \tag{2.1}$$

where G is a given fixed set. G.V. Kamenkov /1/ formulated this definition as well as the stability in the small in a finite time interval.

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Definition 2. System (1.1) is stable in the meaning of this definition, when there exists a G for which system (1.1) is stable in the meaning of definition 1. Here, set G is not specified, but is selected from class K.

Lemma 1. There exists for set $G \in K$ a function $W(x) \in C$ that in some neighborhood ϑ of set ∂G^- satisfies the following conditions:

1) $(\forall x \in \vartheta)\ W(x) > 0;$ (2.2)
2) function $W(x)$ is single-valued;
3) $(\exists L = \text{const})\ \| W' \|_{G(\vartheta)} < L;$
4) $(\forall c_1, c_2 \in E_W)\{x \mid W(x) = c_1 \wedge W(x) = c_2\} \subset \vartheta$
$[\text{int}_1 \{x \mid W(x) \leqslant c_1\} \subset \text{int}_1 \{x \mid W(x) \leqslant c_2\} \wedge$
$\text{int}_1 \{x \mid W(x) \leqslant c_1\} \cap \{x \mid W(x) = c_2\} = \varnothing] \Leftrightarrow (c_1 < c_2);$
5) $(\forall y \in \partial G^-)(\forall z \in \partial G^-)\ W(y) = W(z) \overset{Df}{=} c_0;$
6) $(\forall c \in E_W)(\exists R > 0)\{x \mid W(x) = c\} \subset \text{int}_1 S_R;$
7) $(\forall c \in E_W)$ and the set $\{x \mid W(x) = c\} \subset \vartheta$ $\text{int}_1 \{x \mid W(x) \leqslant c\}$ is connected;
8) $(\forall c \in E_W)\ \text{mes}_{R^n}(\{x \mid W(x) = c\}) = 0$.

When the boundary of ∂G^- is specified in the form $g(x) = 0$, where $g(x) \in C^p$, then $W(x) \in C^p$.

Proof. Consider the homothety transformation with the coefficient $k\ (0 \leqslant k \leqslant 1$. Surface ∂G^- then becomes some surface ∂G_k^-. We select a half-line issuing from θ which intersects ∂G^- at point $x = (x_1, \ldots, x_n)$ and surface ∂G_k^- at point $x^k = (x_1^k, \ldots, x_n^k)$ $(\partial G_k^- = \{x \mid g_k(x) = 0\})$.

We juxtapose to each surface ∂G_k^- number $k^p \| x^k \|^2$, defining by the same token some function $W(x)$. This definition implies that $W(x) \in C$ and possesses properties (1)—(8) defined in (2.2).

Let $\partial G^- = \{x \mid g(x) = 0\}$ and $g(x) \in C^p$.

By definition $W\, \partial G_k^- = \{x \mid W(x) = \| x^k \|^2 k^p\}$. The selected half-line can be made to coincide with the positive direction of the coordinate axis X_1 without loss of generality. Then $g(x) = W(x) - (x_1)^2 k^p$, where x_1 is the coordinated of point x on the X_1 axis. Since the property $g_k \in C^p$ is not affected by homothety, hence $W(x) \in C^p$ in the neighborhood of ϑ.

Remarks. 1) The proof of lemma implies that function $W(x)$ may be defined by its level surfaces and an arbitrary function determined on some curve that intersects each level surface at one point only and has the properties (2.2) and is strictly monotonic $- 1°$—$3°$. $8°$.

2) If $G \in K$ and G is convex, there exists a function determinate not only for $\vartheta(\partial G^-)$ but, also, on the whole R^n. This function can be defined so that $W(\theta) = 0$. From this follows the relation between the introduced functions and Liapunov functions.

We denote by $\vartheta_\varepsilon^+(\partial G^-)$ the ε-neighborhood of ∂G^- points outside $\text{int}(\partial G^-)$, i.e. $\vartheta_\varepsilon^+(\partial G^-) \overset{Df}{=} \{x \mid x \overline{\in} \text{int}(\partial G^-) \wedge \rho(x, \partial G^-) < \varepsilon\}$, and $\vartheta_\varepsilon^-(\partial G^-)$ is the neighborhood of points inside $\text{int}(\partial G^-)$, which is similarly defined.

Theorem 1. 1) For system (1.1) to be stable on set $G \in K$ in the meaning of Definition 1 it is necessary and sufficient that there exists function $V(x) \in C$ that satisfies conditions (2.2) on ϑ and

$$(\forall x(t_0) \in \partial G^-)\ (\forall t \geqslant t_0)\quad V(x(t, t_0, x_0)) \leqslant c_0 \tag{2.3}$$

2) If the boundary of ∂G^- can defined in the form $g(x) = 0$, where $g(x) \in C^1$, then $V(x) \in C^1$, and the necessary stability condition in the meaning of Definition 1 is

$$(\forall x \in \partial G^-)\quad dV(x)/dt \leqslant 0 \tag{2.4}$$

3) For stability on G in the meaning of Definition 1 it is sufficient that in some neighborhood of ∂G^- function $V(x)$ satisfies one of the following conditions:

$$\begin{aligned}(\forall x \in \vartheta_\varepsilon^+(\partial G^-))\quad dV(x)/dt \leqslant 0\\ (\forall x \in \vartheta_\varepsilon^-(\partial G^-))\quad dV(x)/dt \leqslant 0\end{aligned} \tag{2.5}$$

Proof. Lemma 1 implies the existence of function $V(x)$ that satisfies conditions (2.2) and such that

$$G = \{x \mid V(x) \leqslant c_0\}, \qquad \partial G^- = \{x \mid V(x) = c_0\}$$

In this case (2.3) is the necessary and sufficient condition of stability in the meaning of Definition 1. Let us prove statement 2) of the theorem. By virtue of Lemma 1 $V(x) \in C^1$. If we now assume that (2.4) is not true, while the system (1) is stable in the meaning of Definition 1, then there must exist an $x_0 \in \partial G^-$ such that $V^{\cdot}(x_0) > 0$. By virtue of continuity of $V^{\cdot}$ that condition is satisfied in some neighborhood of x_0. Hence the trajectories that begin at points of ∂G^- where $V^{\cdot}(x) > 0$ leave the set $\text{int}_1(\partial G^-)$. Since this is impossible, (2.4) is satisfied. The last statement of the theorem is obvious.

Note that, when the equality in (2.4) is satisfied on a denumerable set, then by virtue of the second condition (F) the system is stable in the meaning of Definition 1.

Remarks. 3) Condition (2.4) is necessary and sufficient for /stability/ of linear systems and sets $G \in K$ bounded by one of second order surfaces.

4) If system (1.1) is stable in the meaning of Definition 1 and condition $(\forall x \in \vartheta_\varepsilon^+(\partial G^-) \setminus \partial G^-)\ dV(x)/dt > 0$ (or $dV/dt > 0$, $\forall x \in \vartheta_\varepsilon^-(\partial G^-) \setminus \partial G^-$) is satisfied, then ∂G^- consists of α limit points in $\vartheta_\varepsilon^+(\partial G^-)$ (ω-limit). A similar statement holds for some connected subset P of the boundary of ∂G^-, if the integral curves of system (1.1) do not issue $\vartheta_\varepsilon^+(P)$.

5) For system (1.1) to be stable in the meaning of Definition 2 it is necessary and sufficient that there exists a $G \in K$ and function $V(x)$ that satisfies Theorem 1 on G.

Various sufficient conditions of stability in the large were obtained in /1 – 9/ using Liapunov functions. In investigations of stability in the large one is confronted, as a rule, with the question of existence of functions that can be used for such investigations. The formulated above necessary conditions of stability may be considered a transformation of the theorem of second Liapunov method for problems of stability in the large. Conditions (2.2) of Lemma 1 used in Theorem 1 define a class of functions used in stability investigations.

Let $G \in K$ and G be convex, then using the Lebesgue theorem /12/ on differentiability of monotonic functions it is possible to show that function $V(x)$ that satisfies (2.2) is differentiable almost everywhere (mes_{R^n} is the measure of set, where $V^{\cdot}$ is indeterminate, is zero).

Consider the qualitative pattern of unstable integral curve behavior.

We say that the integral curve $x(t, t_0, x_0)$ of system (1.1) passes over surface ∂G^-, if $(\exists t_1, t_2) t_0 \leqslant t_1 < t_2 \leqslant \infty\ (\forall \tau \in [t_1, t_2))\ x(\tau, t_0, x_0) \in \partial G^-$.

If, however, $g(x) \in C^1$ and in the case of function V satisfying conditions (2.2) the derivative $V^{\cdot}(x(t, t_0, x_0)$ is zero at point $y \in \partial G^-$, we shall call point y the point of tangency of the integral curve with surface ∂G^-.

We say that the integral curve of system (1.1) emerges from set $\text{int}_1(\partial G^-)$ at point $z_0 \in \partial G^-$ when

$$(\exists t_1, t_2)\ t_0 \leqslant t_1 < t_2 < \infty \quad z_0 = x(t_1, t_0, x_0) \wedge (\forall \tau \in (t_1, t_2))$$
$$x(\tau, t_0, x_0) \overline{\in} \text{int}_1(\partial G^-)$$

We introduce the notation

$$A_0 \overset{Df}{=} \{x \mid x \in \partial G^- \wedge V^{\cdot}(x) = 0\}$$
$$Q_1 \overset{Df}{=} \{x \mid x \in \partial G^- \wedge V^{\cdot}(x) > 0\}$$

We separate qualitatively unstable integral curves in the following three types: 1) whose emergence point is a point of the set Q_1; 2) that in which the integral curve issuing from $\text{int}(\partial G^-)$ and touching ∂G^- departs from $\text{int}_1(\partial G^-)$; and 3) in which the integral curve passes over the surface ∂G^- and emerges from $\text{int}_1(\partial G^-)$.

In the case of unstable integral curves of the second type function $V^{\cdot}$ is positive in the neighborhood emergence point of set $\text{int}(\partial G^-)$, on surface ∂G^- it is zero, and at emergence from $\text{int}_1(\partial G^-)$ it is again positive. In the case of unstable curves of the third type $V^{\cdot}$ is zero on ∂G^- and at emergence from $\text{int}_1(\partial G^-)$ it is positive. The points of emergence from $\text{int}_1(\partial G^-)$ are classified in conformity with the classification of integral curves.

In those cases in which in (1.1) $f(x) \in C^1$ and the boundary is fairly smooth it is possible to reduce the set of points A_0, which may be points of emergence, by introducing in the analysis the second derivative of function V by virtue of system (1.1)

Theorem 2. The sufficient stability condition in the meaning of Definition 1 on set $G \in K, \partial G^- = \{x \mid g(x) = 0\}$, $g(x) \in C^2$ of system (1.1) with $f(x) \in C^1$, is the existence of function $V(x)$ that satisfies (2.2) and the inequality

$$(\forall x \in \partial G^-)\ dV(x)/dt \leqslant 0 \wedge d^2V(x)/dt^2 \neq 0 \tag{2.6}$$

Proof. When conditions (2.6) are satisfied unstable integral curves of the third type are not possible. Suppose that unstable curves of the second type exist. Then $V^{\cdot}$ on integral curves issuing from $\text{int}(\partial G^-)$ is positive in a small neighborhood of the issue point, is zero on set ∂G^-, and then again positive. However it follows from (2.6) that at respective points $V^{\cdot\cdot} > 0 \vee V^{\cdot\cdot} < 0$, and by virtue of the equality

$$(\forall x_0)(\forall t \geqslant t_0) V^{\cdot}(x(t, t_0, x_0)) - V^{\cdot}(x_0) = \int_{t_0}^{t} V^{\cdot\cdot}\, dt$$

the assumed behavior of $V^{\cdot}$ is impossible. The theorem is proved.

Thus, when the assumptions of Theorem 2 are satisfied, the points of emergence from $\mathrm{int}_1(\partial G^-)$ can belong either to set Q_1 or $Q_2 \overset{Df}{=} \{x \mid x \in \partial G^- \wedge V^{\cdot}(x) = V^{\cdot\cdot}(x) = 0\}$ (the necessary condition of instability).

In the cases in which function $V(x)$ is fairly smooth it is possible to solve the problem by introducing in the analysis by virtue of system (1.1) higher order derivatives of V.

Theorem 3. If $(\exists l) f(x) \in C^l \wedge g(x) \in C^l \wedge (\forall r), 2r + 1 \leqslant l$, the relations

$$\begin{gathered}(\forall x \in \partial G^-) V^{\cdot}(x) \leqslant 0 \wedge (\forall x \in B_2) \quad V^{\cdot\cdot}(x) = 0 \wedge \\ (\forall x \in B_3) d^3V(x)/dt^3 \leqslant 0 \wedge (\forall x \in B_4) d^4V(x)/dt^4 = 0 \\ \wedge \cdots \wedge (\forall x \in B_{2r-1}) d^{2r-1}V(x)/dt^{2r-1} \leqslant 0 \\ B_i \overset{Df}{=} \{x \mid x \in B_{i-1} \wedge d^{i-1}V(x)/dt^{i-1} = 0\} \quad (i = 2, 3, \ldots, 2r+1) \\ B_1 \overset{Df}{=} \partial G^-, \quad (\forall x \in B_{2r}) d^{2r}V(x)/dt^{2r} \neq 0\end{gathered} \tag{2.7}$$

are satisfied for function $V(x) \in C^l$ that fulfils (2.2), then system (1.1) is stable on G in the meaning of Definition 1.

If besides the fulfillment of (2.7) the equality

$$(\forall x \in B_{2r}) \; d^{2r}V(x)/dt^{2r} = 0$$

is valid, then in the case of

$$(\forall x \in B_{2r+1}) \; d^{2r+1}V(x)/dt^{2r+1} > 0$$

system (1.1) is unstable on G. If

$$(\forall x \in B_{2r+1}) \; d^{2r+1}V(x)/dt^{2r+1} < 0$$

system (1.1) is stable on G in the meaning of Definition 1.

Proof of the theorem is similar to that of Theorems 1 and 2 using the method of mathematical induction.

Thus, when $f(x)$ and G are fairly smooth, the necessary condition of instability is the continuity of one of sets Q_j which for even j is defined by the relations

$$Q_j \overset{Df}{=} \{x \mid x \in \partial G^- \wedge V^{\cdot}(x) = V^{\cdot\cdot}(x) = \ldots = d^jV(x)/dt^j = 0\}$$

and for odd j by

$$Q_j \overset{Df}{=} \{x \mid x \in \partial G^- \wedge V^{\cdot}(x) = V^{\cdot\cdot}(x) = \ldots = d^{j-1}V(x)/dt^{j-1} = 0 \wedge d^jV(x)/dt^j > 0\}$$

3. Consider the problem of constructing a set on which system (1.1) is stable in the meaning of Definition 1. Let us, first, assume that the outer boundary structure is of the form $g(x, a)$, $g(x, a) \in C^1_{x,a}$, $a \in \Omega \subset R^l$.

We assume the set of parameters Ω to be bounded and compact. We construct in some finite domain D_0 $(\theta \in D_0)$ set $G \subset D_0$, for which $(\exists a_1 \in \Omega)\, \partial G^- = \{x \mid g(x, a_1) = 0\}$ and system (1.1) is stable on G in the meaning of Definition 1 (a particular case of stability on D_0) in the meaning of Definition 2. Investigations in Sect.2 imply the existence of the class of functions $V(x, a) \in C^1_{x,a}$; $a \in \Omega$ $(f(x) \in C^1)$, that satisfy (2.2) and enable the investigation of sets stability bounded by $\partial G^- = \{x \mid g(x, a) = 0,\ a \in \Omega\}$.

Let furthermore the inequality

$$(\exists a_0 \in \Omega)(\exists G(a_0) \subset D_0)(\forall x \in \partial G^-(a_0)) V^{\cdot}(x, a_0) < 0 \tag{3.1}$$

be satisfied.

Then by virtue of continuity of $V^{\cdot}(x, a)$ there exist neighborhoods of a_0 and $\vartheta(\partial G^-)$ where condition (3.1) is satisfied.

The algorithm of solution of the stability problem in the meaning of Definition 1 on a selected set is given below. The problem of selecting G is solved by computer by known numerical methods, for example, variation over a fairly small grid covering Ω, random search on set Ω, random search with adaptation, etc. The number of iterations of the algorithm essentially depends on the ratio of measure (mes_{R^l}) of the set of parameters a where (3.1) is satisfied, to the common measure of Ω, and the quantity $\mathrm{mes}_{R^n}(\vartheta(\partial G^-))/\mathrm{mes}_{R^n}D_0$.

We present the algorithm of stability analysis in the meaning of Definition 1 on a selected set $G \in K$. We assume that $g(x) \in C^2$ and $f(x) \in C^1$. In this case there exists function $V(x) \in C^2$ that can be used for stability investigation. We assume that condition (2.4) is satisfied on set ∂G^- and that set A_0 is continuous. Otherwise investigation of stability in the meaning of Definition 1 is finished. Let us compute the quantity $V^{\cdot\cdot}$ on A_0. If set Q_2 is denumerable, system (1.1) is stable on G in the meaning of Definition 1. If the integral curves originating on Q_2 do not emerge from $\mathrm{int}_1(\partial G^-)$, then (1.1) is stable on G in the meaning of Definition 1. The last condition can be checked by random search on a computer using the following algorithm.

Without imposing additional conditions on $V^{\cdot}$ and $V^{\cdot\cdot}$ we assume that points Q_2 are equivalent as regards the emergence from set $\mathrm{int}_1(\partial G^-)$. The test of validity of this hypothesis is given below.

Let us select on set Q_2 a random point x_0 with uniform probability density distribution over Q_2. We integrate (1.1) over some time interval $[t_0, T]$ for $x(t_0) = x_0$ until the integral curve emerges either from Q_2 or from $\mathrm{int}_1(\partial G^-)$. In the latter case the unstable trajectory has been found, and the algorithm is completed.

Definition 3. We call system (1.1) stable with a probability not lower than p on set $G \in K$, if the existence at the initial instant of arbitrary point $x_0 \in G$ implies that $(\forall t \geqslant t_0)\ x(t, t_0, x_0) \in \mathrm{int}_1(\partial G^-)$ with a probability not lower than p.

This definition is close to that given in /13/. A probable stability does not violate the mechanical meaning attributed to the stability concept. Indeed, a projected real object has a determinate reliability, i.e. the probability of proper functioning of its subsystems, hence a definite probability of existence as an object with required properties. Consequently, if the latter probability is lower than the probability of stability, then such system may be considered as satisfying practical requirements.

Let us substantiate the validity of investigation of stability of systems (1.1).

Lemma 2. If system (1.1) is unstable in the selected set $G \in K$ with a smooth boundary ∂G^-, there exists a connected subset of emergence points such that $\mathrm{mes}_{R^{n-1}} Q > 0\ (n > 1)$.

Proof. By virtue of conditions (F) only the following cases are possible for each maximal simply connected subset of emergence points Q: 1) the whole of ∂G^- consists of emergence points; 2) the boundary of set ∂Q consists of emergence points of the third type, and 3) emergence points of the third type are absent in the neighborhood of boundary ∂Q. In the second case the set Q is closed, while in the third it is open. Hence Q is measurable.

Let us assume that a statement opposite to that of the lemma holds, i.e. that $\mathrm{mes}_{R^{n-1}}Q = 0$. This implies the nonexistence of any arbitrary non-empty neighborhood of space R^{n-1} in set Q /14/. Consider the phase flux Φ issuing from Q. By virtue of the above assumption there exists in an arbitrarily small neighborhood of Q points lying between ∂G^- and Φ. Integral curves from the neighborhood of $Q \subset \partial G^{-1}$ cannot pass through these points and, owing to the continuity of $f(x)$, neither the integral curves originating outside $\mathrm{int}_1(\partial G^-)$ can pass through those points.

Thus in a fairly small neighborhood of Q there are points through which no integral curves can pass, which is not possible in the case of dynamic systems (1.1) /11/. Hence $\mathrm{mes}_{R^{n-1}}Q > 0$.

Using the assumption of equivalence of points of Q_2 and applying the uniform probability density over Q_2, after the i-th step of the algorithm one should expect that $\mathrm{mes}_{R^{n-1}} Q / \mathrm{mes}_{R^{n-1}} Q_2 = 1/i$ /15/, 16/ (by virtue of Lemma 2 system (1.1) with $\mathrm{mes}_{R^{n-1}} Q_2 = 0$ is stable on G in the meaning of Definition 1). As $i \to \infty$, either $\mathrm{mes}_{R^{n-1}} Q = 0$, or an unstable integral curve is obtained. It follows from Lemma 2 that in the first case the system is stable. Thus the algorithm yields a solution of the problem that is convergent with respect to probability, and the estimate of probability of stability in G in the meaning of Definition 1 is, after the i-th step, equal $1 - 1/i$.

In the case of incompressible systems for which

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} = 0$$

the above estimate of probability is improved. Indeed, in the theory of stability initial perturbations are assumed independent and equivalent, hence appearance of any perturbation $x_0 \in \mathrm{int}_1(\partial G^-)$ is equaly probable for $x(t_0)$. The appearance of an unstable integral curve can occur with a probability not greater than $1/i^2$.

The incompressibility condition is satisfied in the particular case of Hamiltonian systems on the basis of the Liouville theorem. Estimates of this kind are also valid for certain algorithms that uniformly select the random numbers $x_0 \in \mathrm{int}_1(\partial G^-)$.

In the more general case $f(x) \in C$, $g(x) \in C^1$ algorithms similar to those for A_0 can be applied.

Examples. 1) Let $n = 2$ and ∂G^- have the form of a square with sides equal a, and symmetric relative to θ

$$\partial G^- = \begin{cases} |x| = a & \text{for } |x| \geqslant |y| \\ |y| = a & \text{for } |y| > |x| \end{cases}$$

The selected region G is a particular case of regions widely used in investigations of stability in a finite time interval /2/.

Stability in the meaning of Definition 1 in region G can only be judged in the case of continuous functions $V(x)$. Continuously differential, and even more so, analytic functions $V(x)$, which could be used for evaluating the stability of an arbitrary mechanical system in a selected set G, do not exist.

Function V can be selected, for instance, in the form

$$V(x, y) = \begin{cases} |x| & \text{for } |x| \geqslant |y| \\ |y| & \text{for } |y| > |x| \end{cases} \quad \text{or } V = \begin{cases} e^{|x|} - 1 & \text{for } |x| \geqslant |y| \\ e^{|y|} - 1 & \text{for } |y| > |x| \end{cases}$$

Note that, when everywhere, except on the bisectors of coordinate angles, condition (2.5) is satisfied, then (1.1) is stable in the meaning of Definition 1 by virtue of the second condition (F). Continuously differentiable functions that can be as close as desired to the considered here functions V do exist, but their derivation and assessment of correctness of their application require special conditions for yielding "wide" sufficient stability conditions.

2) Let $G = \{x \mid x^T \sigma x \leqslant c_0\}$, where $(\forall x)\ x \neq \theta\ x^T \sigma x > 0$ and $\sigma_{ij} = \sigma_{ji}\ (\forall i,\ j = 1, 2, \ldots, n)$. The necessary conditions of stability in the meaning of Definition 1 for system (1.1) are of the form

$$f^T(x)\,\sigma x \,|_{x^T\sigma x = c_0} \leqslant 0$$

If this condition is not satisfied, the system is clearly unstable in the selected set G.

If the nonempty set

$$A_0 = \{x \mid f^T(x)\,\sigma x \,|_{x^T\sigma x = c_0} = 0\}$$

exists, the sufficient condition of stability in the meaning of Definition 1 for $f(x) \in C^1$ is

$$\sum_{l, i, j=1}^{n} \left(\frac{\partial f_i(x)}{\partial x_l} \sigma_{ij} x_j + f_i(x)\,\sigma_{il} \right) \Big|_{x \in A_0} \neq 0$$

3) Consider the application of the proposed numerical methods on the example of stabilization in the field of central force of a material point circular motion controlled by the reaction force /17,18/. We write the equations of perturbed motion in the form

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -\frac{\mu_0}{(r_0 + x_1)^2} + \frac{[(\mu_0 r_0)^{1/2} + x_3]^2}{(r_0 + x_1)^3} + bu$$

$$\frac{dx_3}{dt} = \frac{r_0 + x_1}{r_0} u, \quad u = r_0 c_\varphi \frac{d \ln m}{dt}, \quad b = \frac{c_r}{c_\varphi r_0}$$

$$x_1 = r - r_0, \quad x_2 = r^{\cdot}, \quad x_3 = r^2 \varphi^{\cdot} - (\mu_0 r_0)^{1/2}, \quad \mu_0 = \gamma M$$

where r and φ are polar coordinates of the point, r_0 is the radius of the unperturbed circular orbit, γ is the universal gravitational constant, M is the mass of the center of attraction, c_r and c_φ are projections of relative velocity of the separated particle on the directions of the radius and transverse direction, respectively, and m is the particle mass.

Liapunov function and stabilizing control of the form

$$V(x, \lambda, \mu) = x_2^2 + \frac{[(\mu_0 \mu r_0)^{1/2} + x_3]^2}{(r_0 + x_1)^2} - \frac{2\mu_0 \mu}{r_0 + x_1} + \frac{\mu_0 \mu}{r_0} - \frac{2x_3 (\mu_0 \mu r_0)^{1/2}}{r_0 + x_1} + \lambda x_3^2$$

$$u(x, \lambda, \mu) = -\left(\frac{\partial V}{\partial x_2} b + \frac{\partial V}{\partial x_3} \frac{r_0 + x_1}{r_0} \right)$$

where derived in /18/ using the arbitrary positive parameter μ equal unity.

Application of the proposed above algorithm of random search in the case of $\mu = 1$, $r_0 = 150$, $\mu_0 = 0, 1$, $b = 0.8$, $\lambda = 4.1$ showed that the region of attraction includes with a probability of $8 \cdot 10^{-3}$ the set $\{x \mid V(x) \leqslant 1.532\}$. In the class of Liapunov functions $V(x, \lambda, \mu)$ with $\lambda \in [4, 24]$, $\mu \in [1, 10]$ and $r_0 = 10$ parameters λ and μ were obtained with the probability of $3 \cdot 10^{-3}$, for which the measure of the attraction region is maximum, namely, when $\lambda^* = 8$, $\mu^* = 5$, the measure $\mathrm{mes}_{R^3}\{x \mid V(x) \leqslant 0.0259\}$ of the attraction region is equal 0.01774. In a narrower class of functions $V(x, \lambda, 1)$ and $\lambda \in [4, 24]$ the maximum attraction region is equal $0.562 \cdot 10^{-5}$ with the probability of $5 \cdot 10^{-3}$ with $\lambda^* = 23$.

Note that in any set $\mathrm{int}_1\{x \mid V(x) = c\}$ in the attraction region the system is stable in the meaning of Definition 1.

Remarks. 6) If system is stable in the meaning of Definition 1 or 2, it is dissipative /19/.

7) Note that the theorems considered here can be applied in investigations of stability in finite time intervals, for discontinuous systems subjected to continuous perturbations. They can be extended to the case of nonautonomous sets

$$G = G(t)\ (\forall t \geqslant t_0)\ G(t) \in K.$$

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